Appendix A
Cooper Pair Box Hamiltonian

The Cooper pair box consists of a superconducting island coupled to a superconducting ground via a Josephson junction, or two islands connected by a Josephson junction. The Hamiltonian has the same form in both cases. The equivalent circuit is shown in Fig. (4.1). The bias voltage V could be an intentionally applied dc or ac voltage, or a quantum fluctuating voltage associated with a microwave photon field, or it could be a random voltage representing some local charge asymmetry in the vicinity of the island. For simplicity, we will take the voltage to be supplied by an ideal zero-impedance source. Because figuring out the Hamiltonian of a system connected to a power supply can be confusing, we will consider a physical realization of the voltage source as a very large 'buffer' capacitor C_B as shown in Fig. (A.1). In this configuration, the circuit has two islands with corresponding node fluxes, Φ_1 and Φ_2. The charging energy can be determined by writing the Lagrangian for the two flux variables

$$\mathcal{L} = \frac{1}{2}C_\mathrm{B}\dot{\Phi}_2^2 + \frac{1}{2}C_\mathrm{g}\left(\dot{\Phi}_2 - \dot{\Phi}_1\right)^2 + \frac{1}{2}C_\mathrm{J}\dot{\Phi}_1^2. \tag{A.1}$$

Defining

$$\Phi \equiv \begin{pmatrix} \Phi_1 \\ \Phi_2 \end{pmatrix}, \tag{A.2}$$

the Lagrangian can be written

$$\mathcal{L} = \frac{1}{2}\dot{\Phi}^\mathrm{T} C \dot{\Phi}, \tag{A.3}$$

where the capacitance matrix is given by

$$C = \begin{pmatrix} C_\mathrm{g} + C_\mathrm{J} & -C_\mathrm{g} \\ -C_\mathrm{g} & C_\mathrm{g} + C_\mathrm{B} \end{pmatrix}. \tag{A.4}$$

The electrostatic Hamiltonian is now readily expressed in terms of the charges canonically conjugate to Φ

$$H = \frac{1}{2}Q^\mathrm{T} C^{-1} Q, \tag{A.5}$$

where

$$Q \equiv \begin{pmatrix} Q_1 \\ Q_2 \end{pmatrix}. \tag{A.6}$$

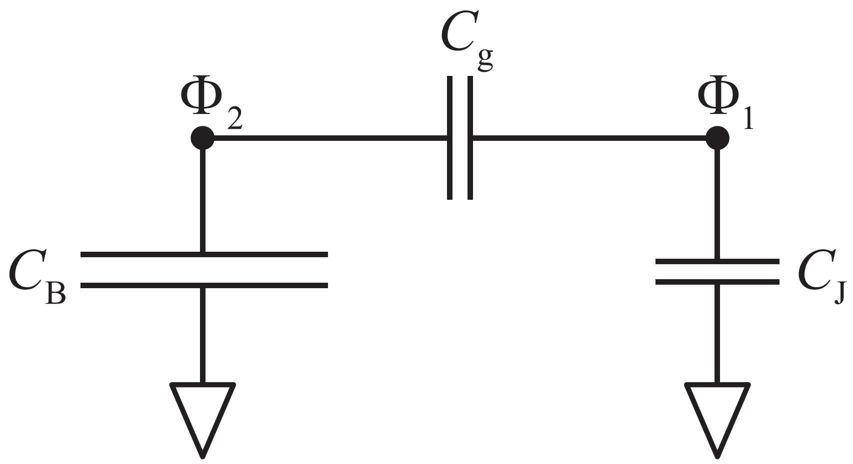

Fig. A.1 Equivalent circuit for determining the electrostatic energy of a Cooper pair box biased by a voltage source represented by a large capacitor C_{B}. The Josephson junction capacitance C_{J} is coupled to the voltage source via the capacitor C_{g}. There are two node fluxes Φ_1 and Φ_2.

The inverse capacitance matrix is given by

$$C^{-1} = \frac{1}{C_{\mathrm{g}}C_{\mathrm{B}} + C_{\mathrm{g}}C_{\mathrm{J}} + C_{\mathrm{B}}C_{\mathrm{J}}} \begin{pmatrix} C_{\mathrm{g}} + C_{\mathrm{B}} & C_{\mathrm{g}} \\ C_{\mathrm{g}} & C_{\mathrm{g}} + C_{\mathrm{J}} \end{pmatrix}. \tag{A.7}$$

This can be simplified by defining the total capacitance to ground for each of the two islands

$$C_{1\Sigma} \equiv C_{\mathrm{J}} + C_{2\mathrm{s}} \tag{A.8}$$

$$C_{2\Sigma} \equiv C_{\mathrm{B}} + C_{1\mathrm{s}}, \tag{A.9}$$

and the two series capacitances

$$\frac{1}{C_{1\mathrm{s}}} \equiv \frac{1}{C_{\mathrm{g}}} + \frac{1}{C_{\mathrm{J}}} \tag{A.10}$$

$$\frac{1}{C_{2\mathrm{s}}} \equiv \frac{1}{C_{\mathrm{g}}} + \frac{1}{C_{\mathrm{B}}}. \tag{A.11}$$

Using these definitions, the inverse capacitance matrix can be written

$$C^{-1} = \begin{pmatrix} \frac{1}{C_{1\Sigma}} & \frac{\beta}{C_{2\Sigma}} \\ \frac{\beta}{C_{2\Sigma}} & \frac{1}{C_{2\Sigma}} \end{pmatrix}, \tag{A.12}$$

where the dimensionless coupling constant is given by

$$\beta \equiv \frac{C_\mathrm{g}}{C_\mathrm{g} + C_\mathrm{J}}. \tag{A.13}$$

The electrostatic Hamiltonian can then be written

$$H = \frac{Q_1^2}{2C_{1\Sigma}} + \beta \frac{Q_2}{C_{2\Sigma}} Q_1 + \frac{Q_2^2}{2C_{2\Sigma}}. \tag{A.14}$$

We define the 'nominal' bias voltage as[1]

$$V_\mathrm{B} = \frac{Q_2}{C_{2\Sigma}}. \tag{A.16}$$

In terms of this we can write the electrostatic Hamiltonian as

$$H = \frac{Q_1^2}{2C_{1\Sigma}} + \beta V_\mathrm{B} Q_1 + \frac{1}{2} C_{2\Sigma} V_\mathrm{B}^2. \tag{A.17}$$

Including now the Josephson junction energy and quantizing we arrive at the full Cooper pair box Hamiltonian

$$H = \frac{\hat{Q}_1^2}{2C_{1\Sigma}} + \beta V_\mathrm{B} \hat{Q}_1 - E_\mathrm{J} \cos \frac{2e}{\hbar} \Phi_1 + \frac{1}{2} C_{2\Sigma} V_\mathrm{B}^2. \tag{A.18}$$

Note that the last term is a constant of the motion and can be ignored.

Defining the dimensionless offset (or 'gate') charge n_g

$$n_\mathrm{g} \equiv -\beta \frac{C_{1\Sigma} V_\mathrm{B}}{2e} \approx -\frac{C_\mathrm{g} V_\mathrm{B}}{2e}, \tag{A.19}$$

(with the latter equality only in the limit of large C_B) and defining the charging energy

$$E_\mathrm{C} \equiv \frac{e^2}{2C_{1\Sigma}}, \tag{A.20}$$

we can also write the Hamiltonian in terms of the integer-valued number operator

$$\hat{n} \equiv \frac{\hat{Q}_1}{2e} \tag{A.21}$$

representing the excess number of Cooper pairs on island one. Its conjugate variable is the relative phase angle for the superconducting order parameter across the junction

$$\varphi = \frac{2e}{\hbar} \Phi_1 = 2\pi \frac{\Phi_1}{\Phi_0}, \tag{A.22}$$

[1] N.B. The *actual* voltage on island two is

$$V = \frac{\partial H}{\partial Q_2} = V_\mathrm{B} + \beta \frac{Q_1}{C_{2\Sigma}}. \tag{A.15}$$

In the limit of of large C_B the actual voltage is fully buffered (i.e., becomes independent of Q_1) and is given by $V \approx V(Q_1 = 0) = V_\mathrm{B}$.

where Φ_0 is the superconducting flux quantum. In terms of this pair of dimensionless charge and phase variables, the Hamiltonian becomes

$$H = 4E_{\mathrm{C}}\left(\hat{n}^2 - 2n_{\mathrm{g}}\hat{n}\right) - E_{\mathrm{J}}\cos\varphi + \frac{1}{2}C_{2\Sigma}V_{\mathrm{B}}^2 \tag{A.23}$$

$$= 4E_{\mathrm{C}}\left(\hat{n} - n_{\mathrm{g}}\right)^2 - E_{\mathrm{J}}\cos\varphi + \frac{1}{2}\frac{C_{1\Sigma}C_{2\Sigma}}{C_{\mathrm{g}} + C_{\mathrm{J}}}V_{\mathrm{B}}^2, \tag{A.24}$$

It is sometimes convenient to work in the (angular) position basis with wave function $\Psi(\varphi)$ and $\hat{n} = -i\frac{\partial}{\partial\varphi}$ being represented by the angular momentum conjugate to the angle φ. In other cases, it is more convenient to work in the angular momentum (charge) eigenbasis and recognize that the operator $\cos\varphi$ term is a 'torque' that changes the angular momentum by ± 1 unit.

Dropping the last constant term in Eq. (A.24) and assuming the voltage bias is fully buffered (large C_{B}), we arrive at Eq. (4.2). We again emphasize that n_{g} is a continuous variable (and often subject to $1/f$ noise), while $\hat{n}$ is integer-valued (i.e., is the angular momentum conjugate to the angular variable φ) and changes by ± 1 each time a Cooper pair tunnels through the Josephson junction connected across C_{J}.

A.1 Cooper Pair Box Coupled to an LC Resonator

Consider the circuit in Fig. (A.2) which shows a Cooper pair box coupled to an LC resonator. For simplicity we will ignore the possibility of a dc bias voltage on the qubit. The Hamiltonian can be immediately written down by analogy with Eq. (A.14)

$$H = H_1 + H_2 + H_{12} \tag{A.25}$$

$$H_1 = \frac{\hat{Q}_1^2}{2C_{1\Sigma}} - E_{\mathrm{J}}\cos\frac{2e}{\hbar}\hat{\Phi}_1 \tag{A.26}$$

$$H_2 = \frac{\hat{Q}_2^2}{2C_{2\Sigma}} + \frac{1}{2L_{\mathrm{B}}}\hat{\Phi}_2^2 \tag{A.27}$$

$$H_{12} = \frac{\beta}{C_{2\Sigma}}\hat{Q}_1\hat{Q}_2. \tag{A.28}$$

Let the eigenfunctions of H_1 obey

$$H_1|j\rangle = \epsilon_j|j\rangle, \tag{A.29}$$

and let us denote the matrix elements of the charge operator in this basis by

$$Q_{jk} = \langle j|\hat{Q}_1|k\rangle. \tag{A.30}$$

This is the analog of the dipole matrix elements of an atom. The LC resonator hamiltonian can be written

$$H_2 = \hbar\omega_{\mathrm{c}}\hat{a}^\dagger\hat{a}, \tag{A.31}$$

where $\omega_{\mathrm{c}} = \frac{1}{\sqrt{L_{\mathrm{B}}C_{2\Sigma}}}$, and the second charge operator can be written following Eq. (2.42)

$$\hat{Q}_2 = -iQ_{2\mathrm{ZPF}}\left(\hat{a} - \hat{a}^\dagger\right), \tag{A.32}$$

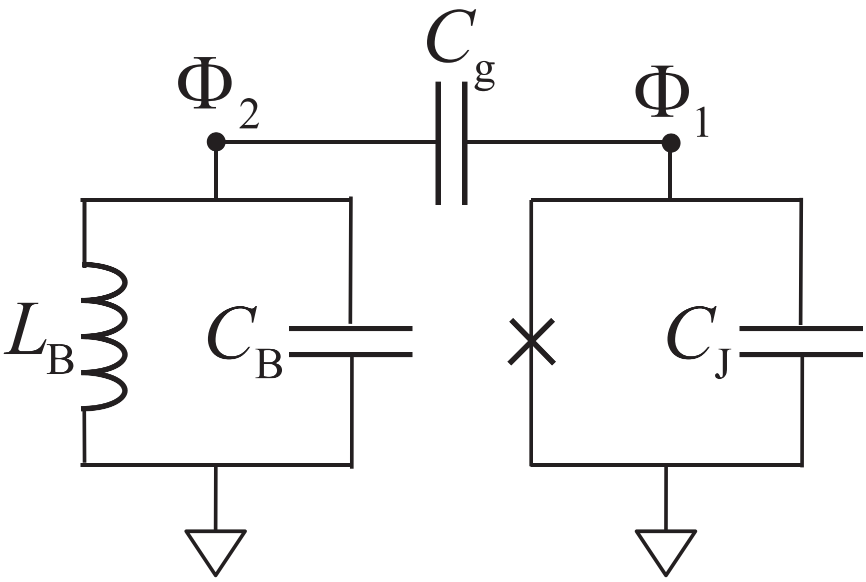

Fig. A.2 Equivalent circuit for a Cooper pair box (without dc voltage bias) capacitively coupled to an LC resonator.

where following Eq. (2.43) we have

$$Q_{2\mathrm{ZPF}} = \sqrt{\frac{C_{\mathrm{B}}\hbar\omega_{\mathrm{c}}}{2}}. \tag{A.33}$$

We now have the full Hamiltonian

$$H = \hbar\omega_{\mathrm{c}}\hat{a}^{\dagger}\hat{a} + \sum_{k=0}^{\infty}\epsilon_k|k\rangle\langle k| - i\frac{\beta Q_{2\mathrm{ZPF}}Q_{jk}}{2C_{2\Sigma}}|j\rangle(\hat{a} - \hat{a}^{\dagger})\langle k|, \tag{A.34}$$

in a form which is convenient for numerical diagonalization.

If the spectrum of the qubit is sufficiently anharmonic, we may be able to restrict our attention to its two lowest states. Projecting our Hamiltonian onto these two states allows us to represent the qubit operators in terms of Pauli spin matrices. Taking the ground state $|0\rangle$ to be represented by spin down $|\downarrow\rangle$ and the excited state $|1\rangle$ to be represented by spin up $|\uparrow\rangle$ we can represent all possible qubit operators within the 2×2 Hilbert space:

$$|0\rangle\langle 0| = \frac{1 - \sigma^z}{2} \tag{A.35}$$

$$|1\rangle\langle 1| = \frac{1 + \sigma^z}{2} \tag{A.36}$$

$$|1\rangle\langle 0| = \sigma^+ \tag{A.37}$$

$$|0\rangle\langle 1| = \sigma^-. \tag{A.38}$$

The Hamiltonian then becomes (dropping an irrelevant constant)

$$H = \hbar\omega_c \hat{a}^\dagger \hat{a} + \frac{\hbar\omega_{01}}{2}\sigma^z - i(\hat{a} - \hat{a}^\dagger)\hbar \left\{ g_{11}\frac{1+\sigma^z}{2} + g_{00}\frac{1-\sigma^z}{2} + g_{10}\sigma^+ + g_{01}\sigma^- \right\} \tag{A.39}$$

where

$$\hbar\omega_{01} \equiv \epsilon_1 - \epsilon_0 \tag{A.40}$$

and

$$\hbar g_{jk} \equiv \frac{\beta Q_{2\mathrm{ZPF}} Q_{jk}}{2C_{2\Sigma}}. \tag{A.41}$$

If the eigenstates of H_1 have a static dipole moment, then the diagonal matrix elements of the charge operator Q_1 will be non-zero. Here we are, for simplicity, ignoring the possibility of a dc bias which produces an offset charge. In this case H_1 has a charge-parity symmetry which guarantees that the diagonal matrix elements of the charge operator Q_1 vanish. We are free to choose a gauge (i.e., choose the arbitrary phases of the eigenstates of H_1) so that $g_{01} = g_{10} = g$ is real. We then arrive at the celebrated Jaynes-Cummings Hamiltonian

$$H = \hbar\omega_c \hat{a}^\dagger \hat{a} + \frac{\hbar\omega_{01}}{2}\sigma^z - i\hbar g(\hat{a} - \hat{a}^\dagger)(\sigma^+ + \sigma^-), \tag{A.42}$$

which, when the rotating wave approximation is justified, further simplifies to

$$H = \hbar\omega_c \hat{a}^\dagger \hat{a} + \frac{\hbar\omega_{01}}{2}\sigma^z - i\hbar g(\hat{a}\sigma^+ - \hat{a}^\dagger\sigma^-). \tag{A.43}$$

We can reduce this to the more familiar expression given in Eq. (6.7) by making a rotation of the spin axes via the unitary transformation

$$\hat{U} = e^{i\frac{\pi}{4}\sigma^z}, \tag{A.44}$$

which yields

$$U\sigma^+U^\dagger = +i\sigma^+ \tag{A.45}$$

$$U\sigma^-U^\dagger = -i\sigma^+. \tag{A.46}$$

and thus finally

$$H = \hbar\omega_c \hat{a}^\dagger \hat{a} + \frac{\hbar\omega_{01}}{2}\sigma^z + \hbar g(\hat{a}\sigma^+ + \hat{a}^\dagger\sigma^-). \tag{A.47}$$